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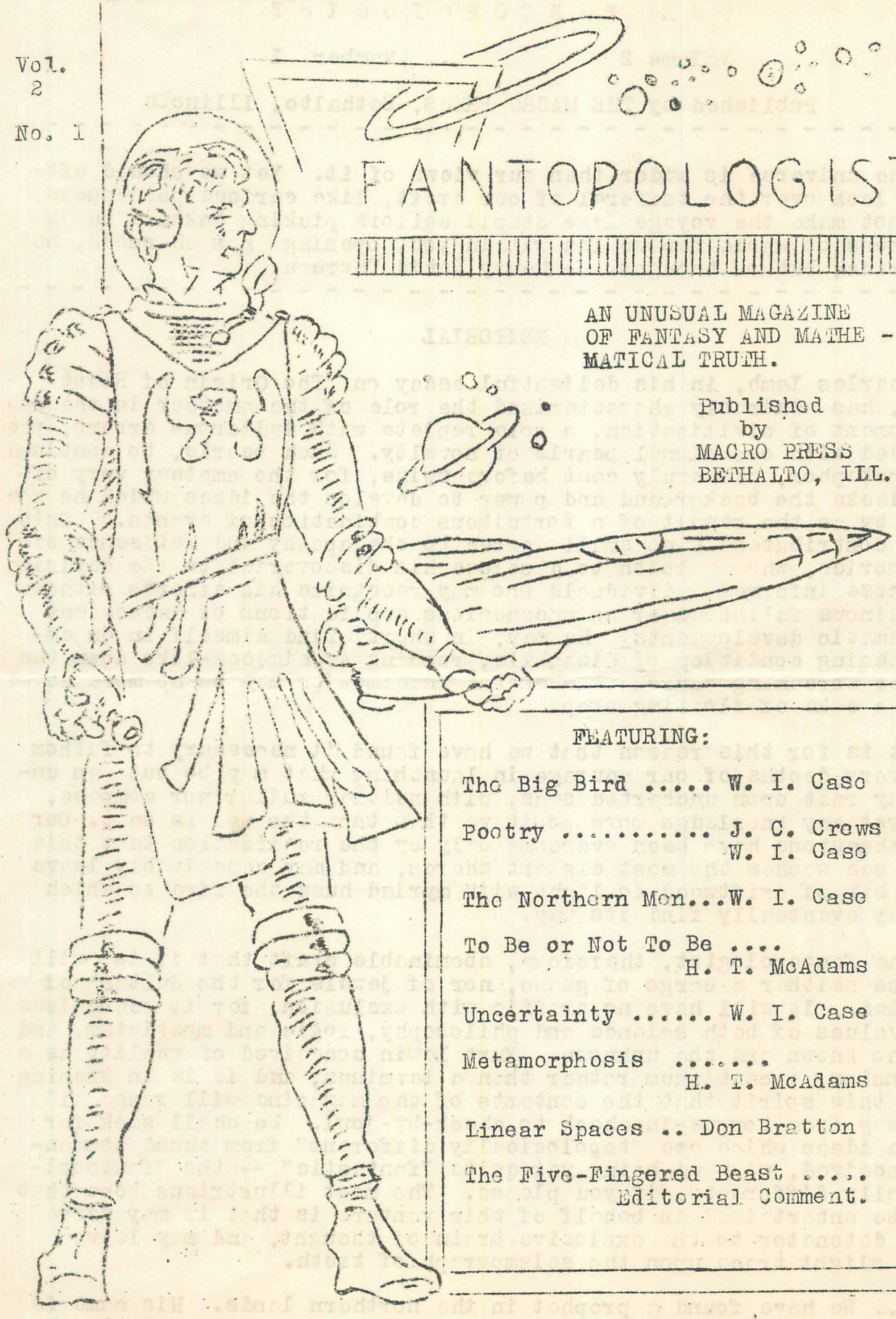

FANTOPOLOGIST

AN UNUSUAL MAGAZINE OF FANTASY AND MATHE - MATICAL TRUTH.

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FEATURING:

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W. I. Case

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THE FANTOPOLOGIST

Volume 2 Number 1

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The universe is wider than our views of it. Yet we should oftener look over the tarferel of our craft, like curious passengers, and not make the voyage like stupid sailors picking oakum. Be a Columbus to whole new worlds within you, opening new channels, not of trade, but of thought. -- Henry David Thoreau.

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EDITORIAL

Charles Lamb, in his delightful essay on "The Origin of Roast Pig", has admirably characterized the role of the amateur in the development of civilization, a role replete with ludicrous errors, yet studded with occasional pearls of novelty. Such pearls, to continue the metaphor, are truly cast before swine, for the amateur very often lacks the background and power to develop the ideas which he has come by as the result of a fortuitous combination of events. This fact contributes in no small degree to the apathy and reticence of the novice, who is loath to disclose his discoveries to the derision of those informed individuals who may recognize his efforts either as heinous fallacies or as unconscious duplications of antecedent systematic developments. He may, in short, find himself in an embarrassing condition of dishabille, running Archimedes-like down the street screaming "Eureka," when the information may be no more novel than a cake of floating soap.

It is for this reason that we have found it necessary to fathom the very depths of our courage in launching what may be but an unworthy raft upon uncharted seas, with neither rudder nor compass, nor yet any knowledge more positive than that the sea is salt. Our apprehensions have been overcome only by the realization that this same sea washes the most distant shores, and may conceivably leave some bit of driftwood to light with myriad hues the fire to which it may eventually find its way.

The Fantopologist, therefore, abominable craft that it is, will refuse neither a cargo of gumbo, nor of jewels for the Sultan of Baghdad. It will have no traffic with exclusion, for it recognizes the values of both science and philosophy, logic and mysticism, and of the known and the unknown. Kurt Lewin conceived of reality as a dimension, a continuum rather than a terminus, and it is in keeping with this spirit that the contents of the magazine will range all the way from tongue-in-cheek to cheek-by-jowl. We shall seek for those ideas which are "topologically different" from those commonly received, without being yet quite "fantastic" -- the "fantopologically different", if you please. The most illustrious hope that may be entertained in behalf of this venture is that it may serve as a detonator to the explosive train of thought, and may leave some slight trace upon the seismograph of truth.

... We have found a prophet in the northern lands. His name is Zar-Kan, and he scoffs at wisdom and foolishness alike. He is the skipper of the craft.

-- H. T. McAdams -- W. I. Case

THE BIG BIRD

By

Bill Case

"I see," murmured Dr. Beutter, who, like many of his kind, may actually have seen much or little.

Little Danny Bartell squirmed in the huge leather chair which almost engulfed him. Danny was embarassed and unhappy. Danny's parents were unhappy, although not a whit embarassed.

"I see," said Dr. Beutter again, making a reflective arch of his two delicate hands on the shiny desk top. "Now, Mr. Bartell, would you say that this, ah -- impression of Danny's is of long standing, or --."

"No, no," said Mr. Bartell. It's just recently that Danny started talking about this bird. Couldn't have been more than two weeks ago. He is so intent, his description so vivid. It's as if he is absolutely convinced that he sees this huge creature out there."

"Um," said Beutter, "Well, of course, these things are not at all unusual in children of Danny's age and temperament. Admittedly he is very consistent in his descriptions, but, again --"

"What are we to do?" interrupted Mrs. Bartell plaintively. "We can't let him continue this way. It's unhealthy."

"Unhealthy, perhaps, Mrs. Bartell, yet not as unhealthy as you fear. This imaginative tendency is a sign of intelligence. It's just that we must distract Danny's alert mind and direct his thoughts into more reasonable channels."

Danny spoke up heatedly. "Aw, you don't know anything. You better watch out, too."

Beutter leaned forward, quizzically intent. "Why, Danny?" he murmured.

"'Cause They don't like people who laugh at them. That's why they're sending this bird. He's gonna grab up the whole world and fly off with it. But he won't hurt me."

"No?" queried Beutter. "Why not, Danny?"

"'Cause I can SEE the bird, and YOU can't. 'Cause I do what They tell me."

"They, Danny? Who are these people you keep talking about?"

"They're NOT people, they're -- well, I dunno -- I never saw 'em. But I know what they want, 'cause they send me messages."

"Ah," said Dr. Beutter.

"Sure. Sometimes I'll be lookin' in my comic book, and when I turn the page there'll be a message there instead of comics. Or when I dig up dirt I sometimes find a rock with a message scratched on it."

"Shades of Charles Fort!" chuckled Beutter, leaning back comfortably.

Danny just frowned and closed his lips firmly as a signal that he would volunteer nothing more. Mr. Bartell looked pained. Mrs. Bartell looked distraught. Beutter leaned forward again and concisely pictured what Danny had repeated to them time and again:

"This bird, bigger than the largest planet, is slowly flying toward Earth from the general direction of Jupiter, and is now somewhere outside the moon's orbit. This bird, at the bidding of certain beings, is intent upon grasping Earth in its talons and flying

away to where "They" reside. Seemingly "They" intend to renovate the planet after which they will turn it over to the bird to plant in the skies again. Believers, like Danny here, will not be harmed but will stay with the planet and take part in its new existence."

Boutter paused while everyone looked glum. He said brightly: "I do not believe the astronomers have noticed anything unusual in the skies. Odd, odd. Can you see the bird from this window, Danny?"

"No," said Danny flatly.

"At what times of the day or night do you see it?"

"Any ol' time I want to except when people that don't believe are around."

"Naturally," said Boutter. "Well --"

Here followed Boutter's reassuring instructions to Danny's parents - instructions calculated to ease the concern of the parents more than to actually divert the boy from his visions. After many sighs and shaking of heads, the Bartells were ushered out of the impressive sanctum of Dr. Boutter. The door closed silently in benediction.

Boutter returned to his desk and sank wearily into the soft chair. He did not yet press the buzzer that set in motion the procedure by which another harried client would be ushered in. He thought: This stupid posturing and posing for the benefit of an ignorant mass of nerves called humanity. Doctor Boutter! As if in this pitiful title there reposed authority, genius, infallibility. And what did he give in return? Nothing: babblings, solemn airs, unctous manners. He smiled sadly, reflecting that you couldn't tell them the truth about themselves. You just couldn't.

He pressed the buzzer.

The door opened. Miss Halloran herded a nervous dowager into the room. Boutter looked at her keenly. Her jittery glance flickered about the room, taking in the expensive paintings, the soft-silent rug, the huge impressive window which looked out upon the great city. Her gaze finally rested on Boutter. She started to speak, but her mouth dropped ajar like the scoop on a power shovel as she realized that the eminent Dr. Boutter was laughing silently, almost hysterically, and was paying her no attention at all, none at all.

Boutter was convulsed, and his laughter was indeed semi-hysterical, for it was obvious that the nervous dowager had seen nothing through the window, and Boutter could only imagine the state of her nerves if she could REALLY see what was out there, this side of the moon now, big as the sky, relentless and imperturbable -- soon to grasp Earth between its incredibly enormous talons.

THE END

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Excerpt from "THE BOOK OF ZAR-KAN"

Of earth and fire are you. Coarse earth calls forth heaven's jagged fire, and in this fury is creation and destruction. Here dwells the Self, borne upward on the flames, pressed into ignominious mud. But the storm is HERE, and not without: Ask not who sends it, for YOU must answer!

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PARTING THEY ONLY SAID THE PSALM WAS DEAR

Should I be analysed only because the window
is an obsession
I saw a window when I was born
my mother saw one in the third month
I was in her womb
let this stump these feeble neurologists
I refuse to count the malady singular
I was born
in an age
subsequently to be known as the age of windows

I will burn behind a window
and be buried beneath a window
and then
there is my art
what window ever obscured more than this

Indeed, in this age of windows
what
indeed ever obscured more than this

-- J. C. Crews

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THE UNIVERSAL HEART

It comes upon a quiet, starry night,
From depths of solemn wonder.
Deep felt, all-encompassing,
It beats like distant thunder.
Ah, where, where to seek?
What depths of feeling plumb;
What search initiate, what probings
Make for silent foxfire, hidden, dumb?
Beat! Beat! You hear it?
'Tis the blood of Being,
Rushing eternally, hear it?
Beyond touch, beyond seeing --
But 'tis there, doubt not;
Though all I've ever known,
The only heart I've ever REALLY heard,
Is here and now - my own.

-- Wm. I. Case

From THE BOOK OF ZAR-KAN

THE NORTHERN MEN

By special permission W. I. Case

At last Pantogar came unto Zar-Kan's house, perched like a gray eagle upon the jutting, wind-swept hill. The journey had been long, the climb strenuous, the thin, chill air burdensome; but now the house of Zar-Kan was reached, and Pantogar rested briefly before saying to the white-haired prophet:

"Zar-Kan, I have heard from some that you are a wise man. I seek your counsel. My people have come upon strange ways, and I know not what to make of it. Though I am chieftain, and though the letter of my commands is obeyed, yet I am aware of an emptiness of spirit. The temples of worship are never empty, yet the people are insincere, I fear. The nights are filled with the sounds of unbridled revelry and naught that I or my guards can do will forestall it for long."

Zar-Kan said: "You seek to know, Pantogar, how you may once more direct your people into the accustomed and moral ways?"

"Yes, Zar-Kan. What is this evil? Never has our community been more prosperous, never has there been more cause for contentment and thanks-giving. But though the gods should be adored and praised for their many kindnesses, there is never a worthy sign of either gratitude or contentment."

"This discontent grows?"

"Yes, in spite of all. It is almost as if the people resent the prosperity and peace that is theirs."

Zar-Kan gazed steadfastly at the blue and gray streaked skies that made frigid patterns beyond the frame of a high window cut into the stone of his house. Long he gazed, and Pantogar finally spoke:

"What would you tell me, Zar-Kan? I have gold to reward you if you will but guide me truly."

Then Zar-Kan said: "They are from the north, eternally from the north, and none shall stand before them. They come from the northern hills and from the northern skies and from that within which is ever cold and clear and eager. They shall come and neither your guards nor priests nor gods shall prevail against them, for they are barbarians and fear not. Their coming is a Siva-coming, a killing and a giving of life. There shall be a mighty fear, a mighty conflict and a mighty destruction, and from the chaos shall rise the epiphenomenon we call civilization. For so it has ever been and will be."

Pantogar said: "Zar-Kan, what is this foolishness you speak? What of the troubles I have related? What of my people?"

Zar-Kan said: "It is falsely written that heat is life and cold is death; I say contrarily. The cold is in their hearts and minds and blood. They turn always toward the warmth, the comfort, the ease, but their essence is of the north and that which is in them compels them to cast aside the warmth and comfort and ease wherever found, however desired, for so it is written. And there

is a great sadness on them and a yearning. There is something of hate in them, too, for so it must be if they seek always destruction; but there is compassion in them, too, for so it must be if their coming is to be a Siva-coming. Pantogar, they hate your laws and your gods and your customs. They will crush your idols and raze your temples and make a mockery of your moral creed. It is their destiny to conquer, and when they have encompassed your world and mine they will scream a challenge at the sun and stars and defy the spirits of heaven and hell to daunt them. For they are masters and leaders and seekers, and the primal urge to conquer burns like a white, cold fire in their breasts. They are beautiful and terrible and lonely, for they may never stop for the caress, the sip of wine; they are a coming and a going, and they may never pause."

Pantogar was deeply troubled. He said: "What is it you seek to tell me, Zar-Kan? Who are these unholy creatures? From whence come they?"

And Zar-Kan said: "They come from every spirit that chafes, from every soul that aspires. In their most cherished possession they see a yoke, in every law a barrier, in every creed a shackle, for within them is the cold fire that would leap and soar; and every temple stands upon a foundation of quicksand. Though time turn full cycle a trillion times, they will not die, nor their fury diminish. You, Pantogar, are a chieftain because your people rose from barbarism and hunger and fear, and made themselves laws to preserve that which they had gained. You are the symbol of those laws. But the wheel continues to turn, and you will go down and your people will go down. In their ignorance and fear they will hate that which they should love, and love that which they should hate."

Pantogar arose fearfully. "Zar-Kan," he said, "surely you know not what you say, for the meaning of your words is madness."

Zar-Kan said: "The meaning of all words is madness."

Pantogar said: "I depart Zar-Kan, and a curse upon you and your foolishness. Much time and effort have I wasted in order to speak with you, and you jest or speak from a disordered mind. Instead of truth you give me fictions. Instead of fact, you give me idle words."

Zar-Kan said: "Truth may not be looked at full-face, for like a god's countenance, it will blind you. We must ever peer above, below, alongside, but never upon it, for if there be madness, that is it."

Pantogar left Zar-Kan's house in great anger and returned no more.

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CLEARLY, SOCRATES!

What is the best that a person can do?
What is the most he can give?
What is his dominant purpose in life?
What is his reason to live?

-- H. P. McAdams

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TO BE OR NOT TO BE: THAT IS IS THE QUESTION

By H. T. McAdams

When, in the course of human events, it becomes necessary to use the word "is", we have sinned the too great sin. Because every statement employing this monosyllable is either a redundancy or a lie. Of course, the disconcerting thing about all this is the fact that any statement at all implies a usage of this verb.

Mathematics, according to L. Wittgenstein, is only a "vast tautology", and the most that it ever manages to say is that "a is a." The apparent variety of mathematical systems is only an unfolding of the numerous symbolic means of accomplishing this end. According to E.T. Bell, better known to science fiction fans as John Taine, this thesis of Wittgenstein's was directly responsible for Carnap's development of the logical syntax of language.

As an example of such tautology, let us consider the well-known maxim: Genius is madness. If indeed the two concepts are identical, then we would require only one term for the two ideas, and would not even need it, since the fact would be obvious at the outset. By finding it necessary or appropriate to make such a definition, we have implied a certain lack of identity between the two equated concepts, and this subtle difference is the only condition which invests the statement with any meaning. Negative statements do not come off any better in the general scheme of things, for to say that a thing is not so-and-so is virtually to imply that it is so-and-so, at least in some respects. The negated statement implies a near-sameness absolutely essential to communication.

In other words, it makes little difference whether we use "is" or "is not", so long as our inferences are in order. It is only by inference that communication is possible, because if two concepts actually are identical, then we have actually said nothing. Perhaps that is why all theories eventually lead to anomaly. With all the inferences and counter-inferences built around "is" and "is not", a balance is very unlikely.

The indicative mode in language, therefore, is actually a subjunctive mode, and "reality" is only a quasi-reality. Charles Fort found it amusing to poke fun at the seriousness with which "science" regards these precious tautologies, and conceived of "reality" as a continuum, and as a unity, about which we have found out nothing, "because there is nothing to be found out." "Existence" is only a quasi-existence, concocted out of the attempt to "be" something, by excluding something else. Fort was anticipated, perhaps, by Hans Vaihinger, who, in his "Philosophy of the As If" attempted to reduce all reality to a set of fictions.

Of course Vaihinger may have been anticipated by Heracleitus when he said that we can not step twice in the same river. But then we can not step in the SAME river even ONCE, because then it is not a river at all; it is a bath-tub.

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UNCERTAINTY: THE CORE OF A NEW PHILOSOPHY

By Wm. I. Case

The student of physics is familiar with the famous Heisenberg Principle of Uncertainty. It is my wish to point out some further implications of this principle, implications which lead inevitably into general philosophic considerations. Particularly, these implications point to a new type of philosophy called cruxism.

The Heisenberg principle may be illustrated in an imaginary experiment in which a physicist attempts to observe the position and velocity of an electron by using a hypothetical super-microscope. In determining its position the velocity of the electron is altered; likewise, when its velocity is determined, its position is altered. Thus it is seen that discovering the fundamental nature of the electron runs into formidable, if not impassable, barriers. The effect of this impasse on the quest for the ultimate secrets of life and matter are obvious.

A generalized uncertainty principle can be applied to psychology and philosophy as well. The psychologist, attempting to determine precisely what motivates certain human actions, is driven by analysis from one stopping-off place backward to the next, and so on, until he is forced to consider matters like electro-neural impulses, brain waves, and other physio-psychical manifestations which are themselves incapable of further analysis, or are based upon unanalyzable substances. In philosophy the question of mind is of the utmost importance. We may ask ourselves, "What is mind?", but in examining the question we find ourselves in the anomalous position of being the observer AND the observed, and completely at a loss to know where the one or the other has jurisdiction or whether either actually exists independently of the other.

In pursuing these thoughts to the utmost, it is likely that the seeker of knowledge will find himself involved in philosophic matters of the profoundest type. Ultimately the seeker must arrive, as I believe at any rate, at a sense of ABSOLUTE UNCERTAINTY. Every quest of whatever type must inevitably shatter on the rocks of paradox and uncertainty. It is this very paradox, this very uncertainty, which we must accept if we are to evolve an adequate science and adequate philosophy.

Picture this situation: a researcher is examining an unclassified object for the purpose of determining its exact nature. He finds -- invariably -- that the nature of the object in toto eludes him because of the effect of his instruments or of the act of observing; yet to eliminate the instruments or the observing act is to eliminate one face of that object's actual nature. The observer rightly concludes that the ultimate nature of the object is therefore uncertain; he cannot logically conclude otherwise.

(Cont. p. 10)

Now here we introduce another factor, this: if the nature of the object is uncertain, then the EXISTENCE of the object is uncertain. That is, the observer, knowing the object in aspect, cannot be sure that the TOTAL, or ultimate nature of the object is not entirely foreign to the observer's limited conception of it. When we look at a star in the night skies, are we actually seeing that star in existence, or ANY star at all? Perhaps not; that star may have exploded and vanished years before we were born; we see merely the light which commenced its journey before the cataclysm. Or perhaps it isn't a star at all; perhaps a gaseous nebula, actually attenuated and misty if viewed at a closer vantage point. Furthermore, if we were to view the nebula from within, we might be quite unaware that it existed, so dispersed would it be. In other words, we put labels on things depending upon the perspective, location, and relation of ourselves to those things. What they actually are -- if they exist -- we simply cannot say for sure.

The observer, as we have already seen, finds himself to be a quantity as unknown as any object under study, and for the same reason. We have this situation then: not only is the nature of the object uncertain; its very existence is uncertain, and, furthermore, the existence of the observer is uncertain. We have arrived, then, at a broader principle than Heisenberg's -- a principle which says: EVERYTHING is uncertain.

While this brief exploration of certain obvious considerations can in no way qualify as thorough justification for the cruxist principle of absolute uncertainty, it is hoped that some few who read this will be tempted to further examination of this principle.

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SPACE OF 3½ DIMENSIONS

By H. T. Mcadams

I know. I should beat a hasty retreat to the nearest booby-hatch and give myself up. But I have it on good authority that it is possible to construct spaces of 2/3 dimensions, 0.4123 dimensions, or even pi dimensions.

In other words, the number of dimensions in a given space does not <u>have</u> to be <u>integral</u>.

Has it ever occurred to you that the inconsistencies attendant in any physical theory, based on three, four, or any other number of dimensions, might be the result of quantization? Maybe our apparent space-time continuum has only 3.999999 dimensions instead of 4. That one part in a million might mean all the difference.

Any wild hairs along this line will be nurtured tenderly by none other than Zar-kan himself.

How's about it?

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METAMORPHOSIS, or HOW TO PERVERT A THEORY

By H. T. McAdams

It has been said, and truly, that the theories of one generation are perverted by the next. Such perversions are often enough to make great men turn over in their graves, and this fact offers little solace to contemporary scientists like Einstein and Heisenberg. The mundane purposes which relativity and uncertainty have already been made to serve are sufficient to insure that these men will indeed have a restless time when they have "shuffled off this mortal coil."

The metric process of going beyond the facts is known as extrapolation, and it is a method of the utmost utility to science fiction. Less familiar, perhaps, is the topological process, which, for want of a better term, may be called metamorphosis, a changing of form.

The prefix "meta-", from the Greek, means "beyond." Metaphysics, then, is literally "beyond physics", but it is a topological rather than a metric "beyond." The subject-matter of metaphysics is non-empirical in nature, and to this extent at least, is topologically different from the "observables" of physics.

Elleston Trevor, in The Immortal Error (London, G. G. Swan), quoting an epigram of Hood, asks:

> "What is mind? No matter. What is matter? Never mind. What is the soul? It is immaterial."

In other words, as any semanticist will agree, such questions are meaningless, because they are not capable of being answered by any meaningful operation. Yet Man, being the Artist that he is, demands symmetry and continuity. To find them, he may need to devise a new discipline, a "meta-semantics" if you will, to take him literally "beyond meaning." Such a discipline might well be to epistemology what the philosophy of Nietsche was to ethics, for "beyond good and evil" lies something which we can classify only as "meta-ethics."

The logical positivists are concerned with "meta-languages", which are required in conjunction with the physical languages. When a statement is made in the "object language" concerning a space-time event, there must often be made, in turn, a statement about that statement, by way of the "meta-language." It is obvious that such an approach permits of infinite regression, forming a hierarchy of languages, each one of which is the subject matter for the one above.

Kurt Gödel, in attacking a similar problem in meta-mathematics, demonstrated that this snipe-hunt is more than an emotional frailty. Certain logical systems can not be proved consistent by their own methods of proof. Within a given system will be certain propositions whose validity can not be established by the rules of that system, but which can be seen to be true from a different viewpoint.

(Continued on page 15.)

LINEAR SPACES by Dan Bratton
(Installment 1 of a series.)

One of the recent conquests of mathematics is the notion of "linear space." It combines what is most essential in many fields of geometry, physics, theory of equations, etc. Like most mathematical ideas, once we have it, it is difficult to understand why the notion, being so simple, was not discovered earlier. With true simplicity comes an illusory feeling of "obviousness."

The notion of linear space comes directly from the operation of addition of numbers. Addition is a process which obeys the following laws:

(A 1) $a + (b+c) = (a+b) + c$. (Associative law.)
(A 2) $a + b = b + a$ (Commutative law.)
(A3) For each a, b, we can find c such that $a + c = b$ (Existence of inverses.)

Note that A3 takes into account the existence of negative numbers. For example, if a = 10, b = 4, then c = -6.

In many parts of algebra, the statements make use only of symbols like our a, b, c in the above, and a "plus" sign. Mathematicians have found it convenient to use this fact to advantage. Such statements are organized around a system in which we pretend that we are dealing, not with numbers, but with general, or abstract "elements". We assume simply that our "elements" obey the three laws A1, 2,3 and do not necessarily obey any particular others. This mental twist results in greater generality and simplicity of statement in mathematics. It has blossomed out into abstract algebra, a program which consistently carries out this new "method".

As examples of the wide range of applications of our "generalized addition", we give the following.

1. Displacements of Euclidean space

Let P be a point in space. Consider various displacements of this point. Label a displacement with a single letter. For example, a could stand for a displacement three inches upward, b a displacement 4 inches o to the right, etc. We can define addition of these letters, or displacements. Let a, b be two displacements. Then "a + b" is defined as the total displacement resulting from first performing the displacement b, then performing the displacement a. (See Fig. 1.) It is readily seen that this "addition" of displacements obeys A1 and A2. How about A3? Well, first note that for every displacement b, there is another which we might note as "-b", which is the opposite displacement, i.e., the displacement of the same distance but in the opposite direction. It is convenient to consider "no displacement at all" itself as a displacement, labeled "z", the zero displacement. Then the displacements b, -b have the property that b + (-b) = z. Further, given two displacements a, b, we take c = b + (-a), (which is usually written for convenience simply "b - a"). Then a + b = b. Thus A3 is satisfied.

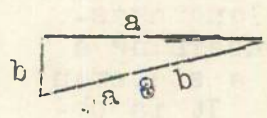

Fig. 1

It should be noted that when we write equations like "a+c = b" we are not talking about the position or positions of the point P, but about the interrelations of certain displacements of P. Thus we have almost forgotten the original subject --P-- and are interested only in certain operations on it. ("Displacements" such as we have been discussing are called "vectors" in physical literature. This is perhaps due to an unwarranted tendency to make the abstract concrete.

2. Additions of Functions

Consider functions y = f(x), y = g(x), etc., x and y being real variables. Given two functions f, g, we can form a new function h, by addition : h(x) = f(x) + g(x). We then write "h = f + g" to show the relation between the functions. This affair is nicely pictured by a graph, which pictures a function as a curve. If f and g are plotted on a graph, then the curve representing h is found by adding the heights of f and g at each point. (See Fig. 2.) Addition of functions is seen to obey A1 and A2, and also A3. The reader might remark at this point, "Of course, we know immediately that this 'addition' of functions will obey all the rules of addition of numbers, because it consists simply of adding the numerical values of the functions." In this course of reasoning, the reader would be dead wrong. To convince himself, he should try multiplying functions. In the latter case, the equation a.c =b, in the given functions a, b cannot always be solved for the unknown function c!

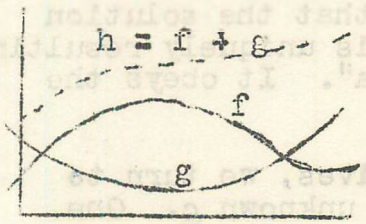

Fig. 2.

Let (f,g) be a pair of functions. When we say that "f equals g" (in symbols, "f = g"), we mean that f(x) = g(x) for all values of x. The "zero function" z is defined as the function z(x) = 0 for all values of x. It has the property that f + z = f for all functions f. It is indeed the only function that has this property. If f is a function, then there is another, written "-f", which is simply the function taking values numerically negative to those of f(x). It has the property that f + (-f) = z. The equation f + h = g, in the given functions f, g and the unknown function h, has the solution h = g + (-f), i.e. h = g - f.

Enough for our two examples. They are sufficient to show the variety of interpretations that can be put upon the laws A1,2,3. Being convinced of their power, we are led to talk about "additive groups", which by definition are simply any set of elements a, b, c, etc., together with an operation "+", which satisfies A1,2,3. The operation in any "additive group" can be written "+" without fear of confusions with numbers.

Let S be an additive group. The equation a + b = a, a being given, b an unknown, has a solution in b, since it is a special case of A3. Denote one solution by b_1, so that b_1 is a definite element and satisfies $a + b_1 = a$. This element b_1 has a deeper property than we might have suspected. For let c be any given element. Then, by A3, there exists a d such that c = a + d. Then $b_1 + c = b_1 + (a + d) = (b_1 + a) + d = c$. Thus the element b_1 is a solution for all such equations as c + b = c, for various values of c! Such an element is called a "zero" element, and will hereafter be denoted "z". We have demonstrated that there exists an element z satisfying a + z = a for all a.

Does an additive group have more than one zero? Does a + b =a have any other solution besides the solution b = b_1 which we supposed found above? Well, suppose b = b_2 is a solution, so that a + b_2 = a. Then by our previous reasoning, b_2 is a zero element, i.e., c + b_2 = c for all c. (We used only the property of b_1 in the above that it was a solution of the equation; thus the conclusions reached apply to any solution.) In particular, this equation must hold for c = b_1. Thus b_1 + b_2 = b_1. But b_1 was also a zero, so, conversely b_2 + b_1 = b_2 ! Thus b_1 = b_1 + b_2 = b_2 + b_1 = b_2., -- that is, b_1 = b_2. Therefore there is one and only one zero, and both our questions are answered.

Now about negatives. Given an element a, the equation a + b=z has, by A3, at least one solution in b. Call it a'. Then a + a' = z, so that a" is another (possibly the same) solution. Consider the expression a + a' + a" = (a + a') + a" = z + a" = a". But, by grouping the first and third terms, we have a + a' + a" = a' + (a + a") = a' + z = a'. Thus a' = a", so that the solution of a + b = z is unique. For each element a, this uniquely resulting element is called the "negative" of a, noted "-a". It obeys the rule a + (-a) = z.

Having demonstrated the existence of negatives, we turn to the entirely general equation a + c = b, in the unknown c. One solution is c = b + (-a), for

$$a + [b + (-a)] = [a + (-a)] + b = z + b = b.$$

Further, this is the unique solution, for if a + c_2 = b, so that c_2 is a solution (possibly the same) then, by adding -a to each member we have

$$(-a) + a + c_2 = (-a) + b,$$

i.e. $$z + c_2 = (-a) + b,$$

or $$c_2 = b + (-a).$$

In an additive group an expression like a + a + a can be abbreviated "3a". In general, if n is an integer positive, negative, or zero, "na" can be appropriately defined. For n positive, "na" is defined as a + a + a + ... + a taken to n terms. For n =0, "0a" is defined to be z. For n negative "na" is defined to be -(-n)a. Under this definition of "multiplication" of the elements of the group by integers we see that

$$na + ma = (n + m)a,$$

$$n(ma) = (nm)a.$$

In school the integers are eventually looked upon as a special case of real numbers. The notion of "linear space" does an analogous thing: multiplication of an element of a group by an integer is broadened into multiplication by real numbers. This is done on a postulational basis.

A LINEAR SPACE is a set S of elements which is an additive group and which at the same time has defined a "multiplication" of its elements by real numbers so as to satisfy certain laws. "(When a is an element of S and r a real number, the "product" is written "ra".)

(L1) For each a in S, 1a = a.

(L2) For each real numbers r,s, and element a of S,

$$(r + s)a = ra + sa.$$

(L3) For each real number r, and each elements a, b of S,

$$r(a + b) = ra + rb.$$

(L4) For each real numbers r, s, and element a of S,

$$r(sa) = (rs)a.$$

The notion of linear space is implicit in our two previous examples. The set of all displacements of the point P in space can be made into a linear space in the following way: Let r be a real number, a a displacement. If a = z, we define ra to be z. Otherwise, let a ≠ z. For r greater than zero, "ra" is defined as the displacement in the same direction as a, but to a distance r times greater. For r = 0, "0a" is z, and for r negative, "ra" is defined as -(-r)a. This definition is readily seen to satisfy L1,2,3,4, so we have a linear space.

Functions can be made into a linear space also. If f is a function f(x) of x, r a real number, then the function "rf" (equals g, say), is defined by g(x) = r.f(x) for all x. The operation forming rf is seen to satisfy L1,2,3,4.

A number of elementary properties of linear spaces are easily developed. If the real number r is actually an integer, application of L1 and L3 gives us the fact that rf has the same meaning as that defined for multiplication of a group element by an integer. Thus 2a = a + a, and (-1)a = -a, as there. There is a simple relation between the real number 0 and the group element z. Namely, (1) rz = z for any real r, and (2) 0a = z, for any element a. These are proved as follows: Let r be a real and a a group element. Then ra = r(a+z) = ra + rz. Subtracting ra gives z = rz. Likewise, ra = (r + 0)a = ra + 0a, so z = 0a. [Next time: Linear subgroups]

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